Formal Logic

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1 Historical Development

Aristotle is credited with codifying and recording the rules of logic, though most of the logic recorded by Aristotle was undoubtedly known to Plato, it is scattered through Plato's works. Aristotle synthesised and compiled this scattered knowledge into a complete whole. Thus we speak of Aristotelian logic. His books where rediscovered by renaissance mathematicians and, along with Euclid, formed a large part of the foundation of subsequent scientific and mathematical investigation.
The rules of logic he described have undergone a number of revisions over the intervening years, particularly by George Boole (1847) who gave the subject a much more algebraic flavour; hence today we talk of Boolean logic. Since then, the definitions have undergone further refinement, some important names in this context would be DeMorgan (1806 - 1871), Georg Cantor (1845 - 1918) and Bertrand Russell (1872 - 1970).
Historically logic is seen as having three basic stages of development.

1. The Greeks
2. Scholastic logic
3. Mathematical logic

The first of these is the logic of Aristotle, in which logical formulas consist of words, subject to the limitations of syntactical and grammatical rules. The second stage was abstracted from normal language, but still had to obey a limited set of rules related to their use in language. The third stage represents a further level of abstraction in which an artificial (mathematical) language was developed to obey certain well determined rules; "the rules of logic". This was the great achievement of George Boole (1847).
The main distinction between the first two stages and the third is that the first two are derived from normal language, whereas in the third there is a purely formal manipulation of logical constructs which then may be translated into normal language. Thus in the first two language underpins the logic, whereas in the third the logic is paramount. We may be able to translate logical statements into English, but the grammar may be awkward. On the other hand we can represent English sentences as logical statements, but we must beware of the fact that often statements in English are ambiguous in a logical sense.
Logic has become one of the cornerstones of modern mathematical theory. In mathematics we have four central objects: Definitions, Axioms, Propositions and Examples. Generally a mathematical work will start with precise definitions of the concepts involved, followed by the axioms. It will then continue with propositions which are proved using logical reasoning. Examples are often provided to illustrate various aspects of the propositions though they are not strictly necessary. In addition more definitions may occur as needed.
Propositions occur in three flavours: Lemmas, Theorems and Corollaries. In general a lemma is a result which is needed in order to prove a theorem, but is not of general interest (in fact some

of the most powerful results are often referred to as lemmas). A Theorem is the main result, that which generalises the concept to be investigated as much as possible. A Corollary is a result which is derived directly from a theorem or was proved incidentally in the course of proving the theorem. Often a result which is used in a practical application will be a corollary of a more general theorem.

2 Statements

Definition 2.1 *A* statement *is a sentence which is either true or false, but not both.*

Note that a statement must be either true or false, it cannot be neither, and it cannot be both. For example the following are statements.

Example 2.2

1. This room is green.

   This is a statement of fact. Provided we have a common understanding of what is green and what is not, then either the room is green or it is not. Note though that we don't care which, only that it must be one or the other.

2. The Goldbach conjecture is true.

   This example talks about a *conjecture*, that is something which has been asserted but is not known to be either true or false. It is not strictly speaking necessary to know what the Goldbach conjecture is, only whether it must be either true or false. In fact this conjecture was stated by Christian Goldbach in 1742:

   > Every even integer greater than two is the sum of exactly two primes.

   The important point is that we do not know the answer, no one has been able to find an even integer which is not the sum of two primes. On the other hand no one has been able to provide a proof that this *must* be so. However, as far as logic is concerned this is irrelevant. The Goldbach conjecture must be either true or false, it cannot be both or neither. Just because we don't know which doesn't change the sentence being a statement.

3. My dog has flees.

   This example is also a statement of fact, either my dog has fleas or it does not. However this is slightly complicated by the fact that I don't have a dog. It doesn't seem reasonable that the mere non existence of the subject (my dog) should suddenly turn this sentence from a statement into something which is not a statement.

   There is actually an ambiguity in the language, we could be implicitly implying that I have a dog, so the statement would more accurately be "I have a dog which has fleas", which is false since I don't have a dog.

   On the other hand we could be saying "All the dogs which I own have fleas", which is vacuously true, since I don't own any dogs. In either case it is a statement since it is definitely true or false. The usual interpretation in logic would be the latter.

We might try to argue that "if I had a dog then it would have flees", or "if I had a dog then it might or might not have flees", but these are actually compound statements consisting of an if…then… conjunction of statements, which we will deal with later.

It is useful to look at some sentences which are not statements.

4. This sentence is a lie.

   This example represents a paradox. If it is true, then by its statement it must be false. On the other hand if it is false it must be true. So we conclude that paradoxical statements are not allowed, since they are neither true nor false.

5. This is a false statement.

   We carry this idea further in this example. In this case the sentence asserts that it is a statement, if it is a statement it is a paradox and so not allowed, if not then it is not a statement, so it is false and hence a statement, oh dear! The problem is easily resolved by realising that we are attempting to assign a definite truth value to something which we have already said is not a statement and hence has ambiguous truth value.

   The essential property which both of these statements have is that of being *self referential,* they talk about themselves. We can see that these types of statements are on the borders of logic. Self referential statements are instrumental in showing Gödel's incompleteness theorem[1].

6. All bus drivers write memos.

   This example is a bit more subtle. The problem here is how we define "All bus drivers". Do we mean those people who are at this moment driving buses? Or perhaps all people who have at any time driven a bus? Or perhaps all members of the bus drivers union? We can see from this that logic is related to the theory of sets. If we had an accepted definition for the set of all bus drivers then we could state categorically whether the sentence was true or false, and hence it would be a statement.

7. $y = x + 3$

   This example suffers from a similar problem, we do not know what x and y are, they could be planets for all we know, for which addition is meaningless. So $y = x + 3$ is not a statement. Even if we know that x and y represent integers or numbers this is still not a statement since we don't have enough information. We would need to know how they relate, consider

   > For every integer x there is an integer y such that $y = x + 3$.

   is a statement, which happens to be true. On the other hand

   > For all integers x and y, $y = x + 3$.

   is also a statement, which happens to be false. But we have no way of knowing which is intended by the original statement $y = x+3$. We will see later on how to deal with statements of this type by using *quantifiers.*

[1] See *Gödel, Escher, Bach* by D. Hofstadter.

3 Compound Statements

In general we use the letters $p, q, r, \ldots$ to represent statements. Note that any given statement has associated with it a truth value T for true or F for false. Though we may not *know* which is associated with a given statement it must be one or the other. With the advent of computers the convention has arisen that 0 means false, and 1 (or any non zero value) means true.

4 Logical Operations

4.1 NOT, AND, OR

There are three basic logical operations which we can apply to statements, they are *not, and, or.* In order to give these operations precise definitions it is useful to introduce the idea of a truth table. In a truth table we tabulate all possible truth values of the input statements. We can show what the output of a compound statement would be for each set of truth values for the inputs. Thus for example, if the only input is one statement, p, and the output is not p we get the table in figure 4.1.

Truth Table for Not p

p	not p
T	F
F	T

If there are two or more inputs, we tabulate every possible combination of values (see figure 4.1).

Truth Table with Two Inputs

p	q	The output goes here.
T	T	X
T	F	X
F	T	X
F	F	X

We may use truth tables to provide definitions of logical connective statements. We may also use them to prove or disprove logical identities.

Definition 4.1

- **NOT**
  The negation of a statement p, denoted not p *or* $\neg p$ *has the opposite truth value as p. The associated truth table is:*

p	$\neg p$
T	F
F	T

- **AND**
  Given two statements p and q we define p and *q, also written $p \wedge q$, to be true only when both p and q are, and false otherwise. Thus the truth table is:*

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- **OR**
  Given two statements p and q we define p or *q, also written $p \vee q$, to be true when either one of p or q is true, or both are true, and false otherwise. Thus the truth table is:*

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Any *not* statement is carried out before any other, unless that part is bracketed. We may come up with complex juxtapositions of statements using *and, or* and *not.* Such statements are called *compound statements.* Thus we may have compound statements such as $p \vee (q \wedge r)$ and $(\neg p \wedge (q \vee r)) \wedge r)$.

4.2 Logical Equivalence

Two compound statements are *logically equivalent* if they have the same truth value for any input. Thus if the columns of the truth tables for the two statements are the same the statements are equivalent. We denote equivalence with the symbol $\equiv$
We can use truth tables to prove the validity of statements. For example we can show that the negation of a negation of a statement is the original statement, that is $\neg(\neg p) \equiv p$.

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

□
We can use this method to prove DeMorgan's laws about the negation of and's and or's.

Theorem 4.2 (De Morgan's Laws)
For any statements p and q

1. $\neg(p \wedge q) \equiv \neg p \vee \neg q$
2. $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Proof:
We give the proof of the first case, the second is left as an exercise. We prove this by showing that the corresponding columns in the truth table are the same. Note that we include some intermediary columns, the columns to compare are denoted by the double lines.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

□
Note that all this has been done without any reference to language, or indeed the meaning of the statements p and q.
There is in fact a linguistic point to be made here. English does not allow the use of brackets, and so to translate $\neg(p \wedge q)$ as 'not p and q' may lead to confusion, as we could mean '$\neg p \wedge q$'. In fact in English we implicitly use DeMorgan's law by saying 'neither p nor q'.

4.3 Tautologies and Contradictions

A compound statement is a *tautology* if all values in the corresponding column of the truth table are T. A compound statement is a *contradiction* if all values in the corresponding column of the truth table are F.
Thus the compound statement $p \vee \neg p$ is a tautology, always true.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

The compound statement $p \wedge \neg p$ is a contradiction, always false.

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Note that a contradiction is a statement which is always false, not a paradox, which is not a statement at all.

5 Logical Equivalences

In the following:
t is a tautology;
c is a contradiction.

Theorem 5.1 (1.1.1) *For any statements, p, q and r the following always hold:*

1	*Commutative laws:*	$p \wedge q = q \wedge p$	$p \vee q = q \vee p$
2	*Associative laws:*	$p \wedge (q \wedge r) = (p \wedge q) \wedge r$	$p \vee (q \vee r) = (p \vee q) \vee r$
3	*Distributive laws:*	$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$	
4	*Identity laws:*	$p \wedge t = p$	$p \vee c = p$
5	*Negation laws:*	$p \vee \sim p = t$	$p \wedge \sim p = c$
6	*Double Negative law:*	$\sim(\sim p) = p$	
7	*Idempotent laws:*	$p \wedge p = p$	$p \vee p = p$
8	*DeMorgans laws:*	$\sim(p \wedge q) = \sim p \vee \sim q$	$\sim(p \vee q) = \sim p \wedge \sim q$
9	*Universal bound laws:*	$p \vee t = t$	$p \wedge c = c$
10	*Absorbtion laws:*	$p \vee (p \wedge q) = p$	$p \wedge (p \vee q) = p$
11	*Negation of t and c:*	$\sim t = c$	$\sim c = t$