Sets - The Element Method

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1 Proving Set Identities - The Element Method

In order to prove the basic set set identities we use the so called element method.
To do this we use the procedural definitions of the set identities:

Definition 1 *Given sets A and B in some universe U and some element $x \in U$:*

- $x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$.
- $x \in A \cup B \Leftrightarrow x \in A \vee x \in B$.
- $x \in A - B \Leftrightarrow x \in A \wedge x \notin B$.
- $x \in A^c \Leftrightarrow x \notin A$.
- $A \subseteq B \Leftrightarrow (\forall x \in A \Rightarrow x \in B)$

To prove a set identity we consider an arbitrary element $x \in U$ and use the properties above, valid forms theorem 1.1.1 to show the result.

Theorem 2 *For any sets A, B and C, $A - B = A \cap B^c$*

Proof: ($\subseteq$) We must show that for every $x \in U$, $x \in A - B \Rightarrow x \in A \cap B^c$.

Let	$x \in A - B$	(Assumption to prove implication)
$\Rightarrow$	$x \in A \wedge x \notin B$	(Definition of setminus)
$\Rightarrow$	$x \notin B$	(Specialization)
$\Rightarrow$	$x \in B^c$	(Definition of B^c)
$\Rightarrow$	$x \in A$	(Specialization)
$\Rightarrow$	$x \in A \wedge x \in B^c$	(Conjunction)
$\Rightarrow$	$x \in A \cap B^c$	(Definition of $\cap$)

($\supseteq$) We must show that for every $x \in U$, $x \in A \cap B^c \Rightarrow x \in A - B$.

Let	$x \in A \cap B^c$	(Assumption to prove implication)
$\Rightarrow$	$x \in A$	(Specialization)
$\Rightarrow$	$x \in B^c$	(Specialization)
$\Rightarrow$	$x \notin B$	(Definition of B^c)
$\Rightarrow$	$x \in A \wedge x \notin B$	(Conjunction)
$\Rightarrow$	$x \in A - B$	(Definition of $A - B$)

Now since $A - B \subseteq A \cap B^c$ and $A \cap B^c \subseteq A - B$, we have $A - B = A \cap B^c$.

Theorem 3 *For any sets A, B and C:*

1. $A \cap B \subseteq A$ *and* $A \cap B \subseteq B$.
2. $A \subseteq A \cup B$ *and* $B \subseteq A \cup B$.
3. $((A \subseteq B) \wedge (B \subseteq C)) \Rightarrow (A \subseteq C)$.

Proof:

1. To show $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

   Assume that some arbitrary element x satisfies $x \in A \cap B$.
   $\Rightarrow x \in A \wedge x \in B$ (Definition of $\cap$).
   $\Rightarrow x \in A$ (Specialization).$\square$

2. We must show that if $x \in A$ then $x \in A \cup B$.

   Let x be an arbitrary, but specific element of A. i.e. $x \in A$.
   $\Rightarrow x \in A \vee x \in B$ (Generalization).
   $\Rightarrow x \in A \cup B$ (Definition of $\cup$).$\square$

3. We must show that if $A \subseteq B$ and $B \subseteq C$ then $x \in A$ implies $x \in C$.
   Assume that $A \subseteq B$ and $B \subseteq C$, and let $x \in A$.
   $\Rightarrow x \in B$ (Since $A \subseteq B$).
   $\Rightarrow x \in C$ (Since $B \subseteq C$). $\square$

Theorem 4 *For any set A in a universe U:*

1. $A \cup \phi = A$ (ϕ is an identity for $\cup$).
2. $A \cap \phi = \phi$ (ϕ is an absorbant for $\cap$).
3. $A \cap A^c = \phi$ and $A \cup A^c = U$.
4. $U^c = \phi$ and $\phi^c = U$.

Proof:

1. ($\subseteq$) We must show that for every $A \subseteq U$, $(x \in A \cup \phi) \Rightarrow (x \in A)$.

Let	$x \in A \cup \phi$	(Assumption to prove implication)
$\Rightarrow$	$x \in A \vee x \in \phi$	(Definition of $\cup$)
$\Rightarrow$	$x \notin \phi$	(Definition of ϕ)
$\Rightarrow$	$x \in A$	(Elimination)

   ($\supseteq$) We must show that for every $A \subseteq U$, $(x \in A) \Rightarrow (x \in A \cup \phi)$.

Let	$x \in A$	(Assumption to prove implication)
$\Rightarrow$	$x \in A \vee x \in \phi$	(Generalization)

   $\square$ $\square$

2. ($\subseteq$) We must show that for every $A \subseteq U,\ x \in A \cap \phi \Rightarrow x \in \phi$.

   Note that the conclusion $x \in \phi$ is always false, for any x. So we must show that $x \in A \cap \phi$ is false for any value of x. We use the rule of contradiction.

Let	$x \in A \cap \phi$		(Assumption to prove implication)
	$\Rightarrow$	$x \in A \wedge x \in \phi$	(Definition of $\cap$)
	$\Rightarrow$	$x \in \phi$	(Specialization)
	$\Rightarrow$	c	(Definition of ϕ)
$\Rightarrow$	$\sim(x \in A \cap \phi)$		(Contradiction)

   ($\supseteq$) We must show that for every $A \subseteq U,\ x \in \phi \Rightarrow x \in A \cap \phi$.

   However, the assumption $x \in \phi$ is false for any value of x.
   Thus the implication is true.

3. To show $A \cap A^c = \phi$

Suppose	$x \in A \cap A^c$	
Then	$x \in A \wedge x \in A^c$	(Definition of $\cap$)
So	$x \in A \wedge x \notin A^c$	(Definition of compliment)
But	there is no such x	(Definition of set)

   So $A \cap A^c = \phi$

   To Show $A \cup A^c = U$.

   ($\subseteq$) If $x \in A \cup A^c$ then $x \in U$, since everything is in U.

   ($\supseteq$) We must show that for every $A \subseteq U,\ x \in A \cup A^c$.

   Given any $x \in U$, either $x \in A$ or $x \notin A$ (definition of set)
   So $x \in A \cup A^c$ (definition of $\cup$).

4. $U^c = \phi$: By the definition of U, here is no element which is not in U.

   $\phi^c = U$

   ($\subseteq$) We must show that for every $x \in U,\ x \notin \phi \Rightarrow x \in U$.

   The statement $x \notin \phi$ is true for every $x \in U$ by the definition of ϕ, so both sides of the implication are always true.

   ($\supseteq$) We must show that for every $x,\ x \in U \Rightarrow x \notin \phi$.

   This is the contrapositive of definition of ϕ.