MTH 210 Midterm Test I Solutions

1. Prove by induction that
$$\sum_{i=2}^{n} i(i-1) = \frac{n(n-1)(n+1)}{3}$$

Proof: (By induction on n)

Base Case Let $n = 2$, $\sum_{i=2}^{2} i(i-1) = 2 \times (2-1) = 2$, $\frac{1}{3}(2(2-1)(2+1) = 2$, so true for $n = 2$.

Inductive Step Let $n = k \geq 2$ Assume true for $n = k$: $\sum_{i=2}^{k} i(i-1) = \frac{k(k-1)(k+1)}{3}$

$$\begin{aligned} \text{Consider } \sum_{i=2}^{k+1} i(i-1) &= \left(\sum_{i=2}^{k} i(i-1)\right) + (k+1)k & \text{(Definition of } \sigma) \\ &= \frac{k(k-1)(k+1)}{3} + (k+1)k & \text{(Inductive Hypothesis)} \\ &= \frac{k(k+1)}{3}((k-1)+3) \\ &= \frac{1}{3}k(k+1)(k+2) \end{aligned}$$

So true for $n = k+1$. □

2. Consider the sequence $a_0 = 1$, $a_n = 3a_{n-1} + 2$, $n > 0$.

   (a) Calculate a_1, a_2, a_3, a_4. Keep your intermediate answers as you will need them for the next part of the question.

$$\begin{aligned} a_1 &= 3 \cdot 1 + 2 & & & &= 5 \\ a_2 &= 3(3+2)+2 &&= 3^2 + 2 \cdot 3 + 2 &&= 17 \\ a_3 &= 3(3^2 + 2 \cdot 3 + 2) + 2 &&= 3^3 + 2 \cdot 3^2 + 2 \cdot 3 + 2 &&= 53 \\ a_4 &= 3(3^3 + 2 \cdot 3^2 + 2 \cdot 3 + 2) + 2 &&= 3^4 + 2 \cdot 3^3 + 2 \cdot 3^2 + 2 \cdot 3 + 2 &&= 161 \end{aligned}$$

   (b) Use iteration to find a formula for the sequence. Simplify your answer as much as possible. In particular, your final answer should not contain sums. Quote any formulas or rules that you use.

$$a_n = 3^n + 2\left(\sum_{i=0}^{n-1} 3^i\right) + 2 = 3^n + 2\left(\frac{3^n - 1}{3 - 1}\right) = 2 \cdot 3^n - 1$$

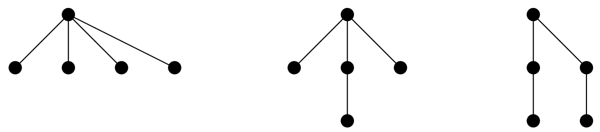
   We have used the formula for the sum of a geometric sequence: $\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$

3. Find a regular expression for the following languages over $\{0, 1\}$

   (a) All string which **do not** begin with a 1.
   $0(0 \vee 1)^* \vee \epsilon$

   (b) All strings in which every odd position is a 1.
   $(1(0 \vee 1))^*(1 \vee \epsilon)$ or $(10 \vee 11)^*(1 \vee \epsilon)$

4. Find all trees with five vertices up to isomorphism.



5. For each of the languages, L, over $\Sigma = \{a, b\}$ defined by the given regular expression:

i. Determine if $\epsilon \in L$; ii. give two nonempty strings in L iii. give two nonempty strings which are in Σ^* but not in L. iv. Give a *short* description of L in English.

(a) $(a^* \vee b^*)$

i. $\epsilon \notin L$. ii. a, b (etc). iii. ab, ba (etc).
iv. Any string consisting entirely of a's or entirely of b's.

(b) $(ab \vee ba \vee bb)(a \vee b)^*$

i. $\epsilon \in L$. ii. ab, ba (etc). iii. ϵ, b, a, aa.
iv. Any string of length at least 2, which does not start with aa.

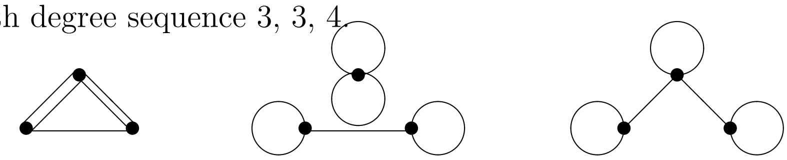
6. For each of the following either explain why the specified graph cannot exist (quote any theorems you use), or draw a graph with the given property.

(a) A graph with degree sequence 3, 3, 3.

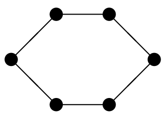
Does not exist. The total degree is $3 + 3 + 3 = 9$, but the total degree must be even.

(b) A graph with degree sequence 3, 3, 4.

One of:

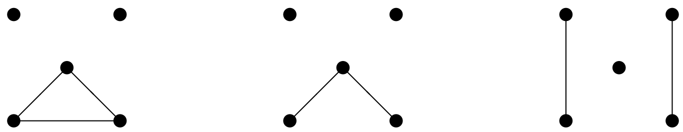


(c) A simple bipartite graph on 6 vertices containing an Eulerian circuit and a Hamiltonian circuit.

$C_6 =$ 

(d) A simple graph with 3 connected components on 5 vertices.

One of:



(e) A binary tree of height 3 with 9 leaves.

Does not exist. There is a theorem that says that $t \leq 2^h$, but in this case $t = 9$ and $h = 3$.

(f) A binary tree with 6 vertices and 3 leaves.

One of: