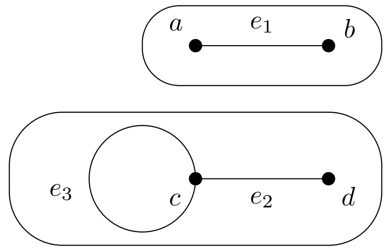
MTH 210 Midterm Test I Solutions

1.



G

For the graph G given above give the following, or explain why the graph does not have the given property. (Quote any theorems you use.)

(a) On the graph circle the connected components of G. *(See graph.)*

(b) What are the endpoints of e_2? c and d

(c) Find a loop in G. e_3, or $\{c, d\}$

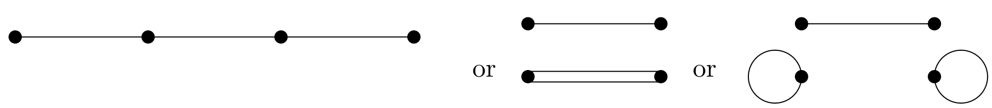
(d) Give the degree sequence of G. $1, 1, 1, 3$

(e) Find a walk from c to d which is not a path. $c\,c\,d$ or $c\,d\,c\,d$, etc.

(f) Find a Hamiltonian circuit in G.
The graph does not contain a Hamiltonian circuit as it is not connected.

2. For each of the following either explain why the specified graph cannot exist (quote any theorems you use), or draw a graph with the given property.

(a) A graph with degree sequence 1,1,2,2.

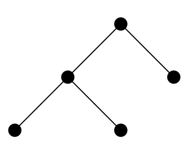


(b) A graph with total degree 11. DNE, total degree is always even. ($TD = 2e$)

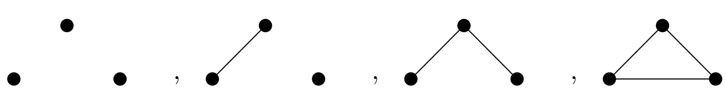
(c) The complete bipartite graph $K_{2,3}$. See notes.

(d) A tree with 7 vertices and 7 edges.
DNE all trees on n vertices have $n - 1$ edges, so a tree on 7 vertices has 6 edges.

(e) A full binary tree with 5 vertices and 3 leaves.



3. Find all simple graphs on three vertices up to isomorphism.



4. Consider the sequence

$$b_0 = 1;$$
$$b_i = 3b_{i-1} + 2 \quad i \geq 1.$$

(a) Find b_1, b_2, b_3 and b_4. Keep your intermediate answers as you will need them for the next part of the question.

$$\begin{array}{rcll} b_1 & = & 3 \cdot 1 + 2 & = 5 \\ b_2 & = & 3 \cdot 3 \cdot 1 + 3 \cdot 2 + 2 & = 17 \\ b_3 & = & 3 \cdot 3 \cdot 3 \cdot 1 + 3 \cdot 3 \cdot 2 + 3 \cdot 2 + 2 & = 53 \\ b_4 & = & 3^4 + 3^3 \cdot 2 + 3^2 \cdot 2 + 3 \cdot 2 + 2 & = 161 \end{array}$$

(b) Use iteration to find an analytic formula for the sequence b_n. Simplify your answer as much as possible, showing your work. In particular, your final answer should not contain sums. Quote any formulas or rules that you use.

$$\begin{array}{rcll} b_n & = & 3^n + \Sigma_{i=0}^{n-1} 3^i \cdot 2 & \text{(From above)} \\ & = & 3^n + \dfrac{3^n - 1}{3 - 1} \cdot 2 & \text{(Sum of a geometric sequence)} \\ & = & 3^n + 3^n - 1 & \text{(Algebra)} \\ & = & \boxed{2 \cdot 3^n - 1} & \end{array}$$

5. Let e_n be the sequence defined by
$$e_0 = 1;$$
$$e_n = \frac{e_{n-1}}{1 + 2e_{n-1}}, \quad n \geq 1.$$

Use weak induction to show that $e_n = \dfrac{1}{2n+1}$ for every $n \geq 0$.

Be sure to lay out your proof clearly and correctly.

<u>Base Case</u> Let $n = 0$, then $e_0 = 1$ and $\frac{1}{2n+1} = 1$. So true for $n = 0$

<u>Inductive Step</u> Let $n = k \geq 0$ and assume $e_k = \frac{1}{2k+1}$.

$$\begin{array}{rcll} \text{Consider } e_{k+1} & = & \dfrac{e_k}{1 + 2e_k} & \text{(Definition of } e_k,\ k+1 \geq 1,) \\ & = & \frac{\frac{1}{2k+1}}{1+2\frac{1}{2k+1}} & \text{(Inductive Hypothesis)} \\ & = & \frac{\frac{1}{2k+1}}{\frac{2k+3}{2k+1}} & \text{(Algebra)} \\ & = & \frac{1}{2k+3} & \text{(Algebra)} \\ & = & \frac{1}{2(k+1)+1} & \text{(Algebra)} \end{array}$$

So true for $n = k + 1$.

6. Use strong mathematical induction to show that postage of n cents can be made up by using some combination of 3 or 7 cent stamps, whenever $n \geq 12$.

In other words:
Show that any integer greater than 11 can be expressed as a sum of multiples of 3 and 7.

Be sure to lay out your proof clearly and correctly.

<u>Base Case</u> $12 = 3 \times 4$, $13 = 3 \times 2 + 7$, $14 = 7 \times 2$.

<u>Inductive Step</u> Let $n = k > 14$ and assume that for every i with $12 \leq i < k$ there exist integers a_i and b_i such that $i = 3a_i + 7b_i$.

$$\begin{array}{rcll} \text{Consider } k & = & k - 3 + 3 & \text{(Algebra)} \\ & = & 3a_{k-3} + 7b_{k-3} + 3 & \text{(Inductive Hypothesis on } k-3,\ k > k-3 \geq 12) \\ & = & 3(a_{k-3} + 1) + 7b_{k-3} \text{(Algebra)} & \end{array}$$

So $a_k = a_{K-3} + 1$ and $b_k = b_{k-3}$ for $k > 14$, and the result is true for $n = k$.