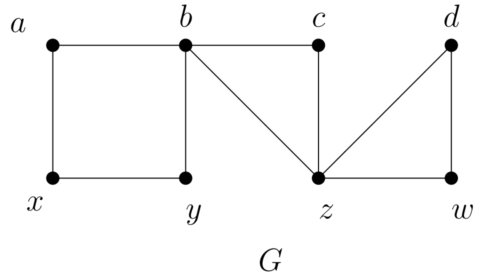
MTH 210 Midterm Test Solutions

1. 

For the graph G given above give the following, or explain why the graph does not have the given property. (Quote any theorems you use.)

(a) Find the order, size, total degree and degree sequence.
Order: 8, Size 10, Total Degree: 20, Degree Sequence 2,2,2,2,2,2,4,4

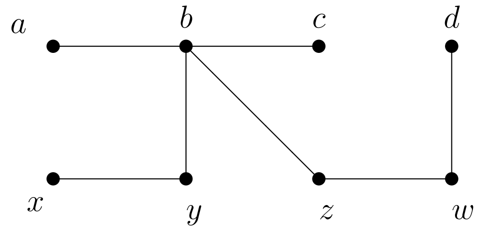
(b) Is G simple? *Yes*

(c) How many connected components does G have? *1 Component*

(d) Find a walk from c to d which is not a path. $czczd$ - There are many possible answers.

(e) Find an Eulerian circuit in G. $baxybczwdzb$ - There are other possible answers.

(f) Draw a spanning tree of G.

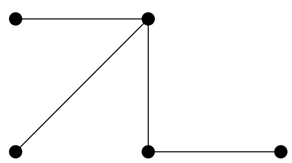


One possible solution.

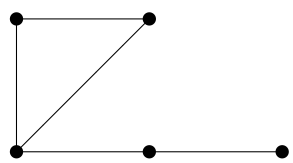
2. For each of the following either explain why the specified graph cannot exist (quote any theorems you use), or draw a graph with the given property.

(a) A simple graph with degree sequence $1, 1, 2, 2, 3$.
Does not exist. The total degtree is odd *or* There are an odd number of vertices of odd degree.

(b) A simple graph with degree sequence $1, 2, 2, 2, 3$.



OR



(c) A bipartite cycle (Recall: a cycle is a simple circuit).
Any even order cycle.

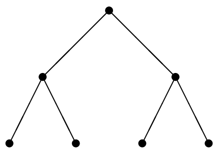
(d) The path of length 4, P_5 and its complement $\overline{P_5}$. *Note* $P_5 \cong \overline{P_5}$.

AND

(e) A 3−regular graph on 4 points. K_4

(f) A full binary tree with 6 vertices and 4 leaves. *Does not exist.*
In a full binary tree, if the number of leaves is $n-1$, then the order is $2n+1$. See Theorem 11.5.5.

(g) A binary tree with height 2 and 3 leaves



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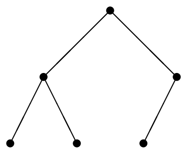
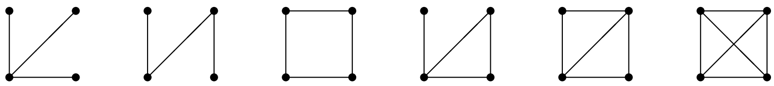


3. List all simple connected graphs on 4 points, up to isomorphism.



4. In the following f, g and h are functions and n is a variable. Answer true or false in each case by circling the appropriate answer. Note that marks may be deducted for incorrect answers.

(a) **F** (b) **T** (c) **T** (d) **F** (e) **T** (f) **T** (g) **F** (h) **T**

5. Consider the sequence

$$\begin{aligned} a_1 &= 2; \\ a_n &= a_{n-1} + n + 1, \quad n > 1. \end{aligned}$$

(a) Find $a_2,\ a_3,\ a_4$ and a_5. Keep your intermediate answers as you will need them for the next part of the question.

$$\begin{aligned} a_2 &= 2+2+1 &&= 5 \\ a_3 &= 2+2+1+3+1 &&= 9 \\ a_4 &= 2+2+1+3+1+4+1 &&= 14 \\ a_5 &= 2+2+1+3+1+4+1+5+1 &&= 20 \end{aligned}$$

(b) Use iteration to find an analytic formula for the sequence a_n. Simplify your answer as much as possible, showing your work. In particular, your final answer should not contain sums. Quote any formulas or rules that you use.

$$\begin{aligned} \text{Guess:} \quad a_n &= \left(\sum_{i=1}^{n} i\right) + 1 + n - 1 \\ &= \frac{n(n+1)}{2} + n \\ &= \frac{n(n+3)}{2} \end{aligned}$$

6. Show by weak induction that 8 divides $3^{2n} - 1$, for every $n \geq 0$.

   Base Case Let $n = 0$, then $3^{2n} - 1 = 0$ and $8 \mid 0$. So true for $n = 0$.

   Inductive Step Let $n = k \geq 0$ and assume $8 \mid 3^{2k} - 1$.
   So $\exists\, m \in \mathbb{Z}$ such that $n = 8m$ (Definition of Divides).

$$\begin{array}{llll} \text{Consider} & 3^{2(k+1)} - 1 & = & 3^2 \cdot 3^{2k} - 1 & (\textit{Algebra}) \\ & & = & (8+1) \cdot 3^{2k} - 1 & \text{(Algebra)} \\ & & = & 8 \cdot 3^{2k} + 3^{2k} - 1 & \text{(Algebra)} \\ & & = & 8 \cdot 3^{2k} + 8m & \text{(Inductive Hypothesis)} \\ & & = & 8(\cdot 3^{2k} + m) & \text{(Inductive Hypothesis)} \end{array}$$

   $3^{2k} + m \in \mathbb{Z}$ (Closure), so $8 \mid 3^{2(k+1)} - 1$ (Definition of Divides). □

7. Let b_n be the sequence defined by $b_0 = 3$, $b_1 = 5$ and

$$b_n = 2b_{n-1} + 3b_{n-2}, \text{ for } n > 1.$$

   Use strong induction to show that

$$b_n = 2 \cdot (3)^n + (-1)^n.$$

   Base Case Let $n = 0$, then $b_0 = 3$ and $2 \cdot (3)^0 + (-1)^0 = 3$. So true for $n = 0$.
   Let $n = 1$, then $b_1 = 5$ and $2 \cdot (3)^1 + (-1)^1 = 5$. So true for $n = 1$.

   Inductive Step Let $n = k > 1$ and assume $b_i = 2 \cdot (3)^i + (-1)^i$ for each $0 \leq i < k$.

$$\begin{array}{llll} \text{Consider} & b_k & = & 2b_{k-1} + 3b_{k-2} & \text{(Definition of } b_n\text{)} \\ & & = & 2(2 \cdot (3)^{k-1} + (-1)^{k-1}) + 3(2 \cdot (3)^{k-2} + (-1)^{k-2}) & \text{(Inductive Hypothesis)} \\ & & & & \text{(Note } k-1, k-2 \geq 0 \text{ since } k > 0\text{)} \\ & & = & 2 \cdot (3)^{k-2}(6+3) + (-1)^{k-2}(-2+3) & \text{(Algebra)} \\ & & = & 2 \cdot (3)^k + (-1)^{k-2} & \text{(Algebra)} \end{array}$$

   So true for $n = k$. □